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STEADY STATE OF A NON-LINEAR EVOLUTIONARY EQUATION,(U)

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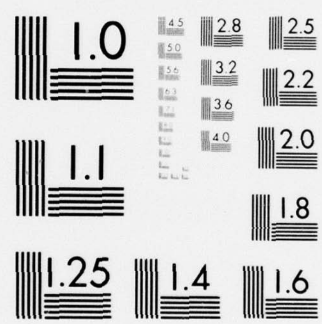

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Steady states of a class of nonlinear evolutionary equations with inhomogeneous (source) terms are investigated. First, the question of when the initial value problems corresponding to the evolutionary equations possess a steady state is discussed. Then, for a wide class of cases, we show that the steady state of the inhomogeneous problem is the same as the steady state of another problem without sources, but with different initial data. A direct proof of this result is given for the one-phase Stefan problem. In this case we also analyze the extent to which the conditions under which the general theorem is proved can be relaxed. An application shows that a free boundary problem in anodic smoothing is equivalent to a steady state one-phase Stefan problem.

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STEADY STATE OF A NONLINEAR EVOLUTIONARY EQUATION

by

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1. Introduction

In a recent paper (Ref. 2), Brezis, Berger, and Rogers have presented an algorithm for solving the initial value problem

$$u_t + L\, f(u) = 0,\ x \in \Omega\ ,\ 0 < t < T, \tag{1.1a}$$

$$B\, f(u) = 0,\ x \in \partial\Omega\ ,\ 0 < t < T, \tag{1.1b}$$

$$u(0) = u_o \in L^1(\Omega) \cap L^\infty(\Omega)\ ,\ x \in \Omega\ , \tag{1.1c}$$

in a bounded domain Ω with smooth boundary. f satisfies $f(0) = 0$ and

$$0 \le f(u) - f(v) \le u - v \text{ for } u \ge v. \tag{1.2}$$

$L: D(L) \subset L^1(\Omega) \longrightarrow L^1(\Omega)$ is a closed linear operator whose domain $D(L)$ is dense in $L^1(\Omega)$ and whose associated semi-group

$$S(t) = e^{-Lt}\ ,\ t > 0, \tag{1.3}$$

is contractive in $L^1(\Omega)$ and $L^\infty(\Omega)$. B is a linear operator, such that the solution of

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$$L\,h = \bar{g},\ x \in \Omega, \tag{1.4a}$$

$$B\,h = 0,\ x \in \partial\Omega\ , \tag{1.4b}$$

satisfies, for all $\bar{g} \in R(L)$,

$$\|h\|_{L^1(\Omega)} \leq \alpha\, \|\bar{g}\|_{L^1(\Omega)}\quad , \tag{1.4c}$$

for some $\alpha > 0$.

The version of the algorithm which is used as a basis for deriving the principal theorem of this paper is as follows. Given

$$u^0 = u_0\,, \tag{1.5a}$$

one constructs

$$u^n = F_0^n\ u_0\,,\ n = 1, 2, 3, \ldots, \tag{1.5b}$$

where

$$F_0\ u \equiv u - f(u) + S\ f(u). \tag{1.5c}$$

(We use the symbol "S" for $S(\tau)$ when there is no chance of confusion.) u^n approximates $u(n\tau)$ in the following sense:

$$\|u^n - u(t)\|_{L^1(\Omega)} \longrightarrow 0 \text{ as } \tau = \frac{t}{n} \longrightarrow 0, \tag{1.6}$$

uniformly for t in the interval $[0,T]$ (Ref. 2).

We are interested in the inhomogeneous equation

$$u_t + L\ f(u) = g(t),\ x \in \Omega\ ,\ t > 0, \tag{1.7a}$$

$$B\ f(u) = 0,\ x \in \partial\Omega\ ,\ t > 0, \tag{1.7b}$$

$$u(0) = u_0\ ,\ x \in \Omega\ , \tag{1.7c}$$

particularly in the steady state of (1.7), when one exists. Thus, some enlargement of the results given above is in order. The generalizations required involve the extension of equation (1.6) to the inhomogeneous problem and replacement of the upper time limit T by ∞. In some cases it may also be desirable to drop the restriction that Ω be bounded.

(When Ω is unbounded, we add to (1.1) and (1.7) the asymptotic condition

$$A\ f(u) \xrightarrow[|x| \to \infty]{x \in \Omega} 0, \tag{1.8}$$

where A is a linear operator, and we assume that for any bounded domains

$\mathcal{D} \subset \Omega$ and $\tilde{\mathcal{D}} \subset \Omega$ and function $\bar{g} \in R(L)$ with supp $\bar{g} \subset \mathcal{D}$,

$\exists \;\; \alpha(\mathcal{D}, \tilde{\mathcal{D}}) > 0$ such that the solution of (1.4a,b) and

$$A\,h \xrightarrow[|x|\to\infty]{x\in\Omega} 0, \tag{1.9a}$$

satisfies

$$\|h\|_{L^1(\tilde{\mathcal{D}})} \leq \alpha(\mathcal{D}, \tilde{\mathcal{D}}) \;\; \|\bar{g}\|_{L^1(\mathcal{D})} \quad . \tag{1.9b)}$$

Here we will not give any formal proof that the algorithm of Ref. 2 can be used in these extended circumstances. Instead, we shall assume:

(A1) For $\bar{g} \in R(L)$ and all

$$g(t) = \bar{g} + g'(t) \tag{1.10a}$$

such that

$$\int_0^\infty \|g'\|_{L^1(\Omega)}\, dt < \infty \quad , \tag{1.10b}$$

(1.7) has a solution which approaches the steady state value $\bar{u}$ satisfying

$$L\, f(\bar{u}) = \bar{g},\; x \in \Omega \;; \tag{1.11}$$

(A2) Uniformly for $\tau \in (0, \tau_0]$, the quantity u^n computed by

$$u^0 = u_0 , \tag{1.12a}$$

$$u^{n+1} = F_{g*}\, u^n + \int_0^\tau S(\tau - \xi)\, g'(\xi + n\tau) d\xi \quad , \tag{1.12b}$$

where

$$F_g\, u \equiv u - f(u) + S(\tau)\, f(u) + g\,\tau \tag{1.12c}$$

and

$$g^*(\tau) = \frac{1}{\tau}\int_0^\tau S(t) dt \;\; \bar{g}, \tag{1.12d}$$

approaches in $L^1(\Omega)$, as $\tau n \longrightarrow \infty$, the steady state $u^*(\tau)$ satisfying

$$\frac{I - S(\tau)}{\tau}\, f(u^*(\tau)) = g^*, \tag{1.13}$$

for all g which satisfy (1.10);

(A3) Given any $T > 0$, (1.6) holds uniformly for $t \in [0,T]$, where u^n and u refer to solutions of (1.12) and (1.7), respectively.

We note that the assumptions (A1) - (A3) are not completely independent. For example, (A2) and (A3) obviously imply (A1). In addition, one may derive from (A1) and (A3) a weakened form of (A2) which is expressed as follows:

(A2') For any $\epsilon > 0$ $\exists\, T_o > 0$ and $\tau_1 \in (0, \tau_o]$ such that, for the quantities u^n computed by (1.12) and the steady state $\bar{u}$ of u given by (1.7),

$$\|u^n - \bar{u}\|_{L^1(\Omega)} \leq \epsilon \tag{1.14}$$

when $n\tau \geq T_o$ and $\tau \leq \tau_1$.

We will see in the sequel that (A1), (A2'), and (A3) (and thus (A1) and (A3)) will be sufficient for us to prove our principal results pertaining to the steady state solution of (1.7). To derive (A2') from (A1) and (A3), note first that (1.10b) and (1.11) imply there is a T_o such that for $T \geq T_o$

$$\int_T^\infty \|g'\|_{L^1(\Omega)} \leq \frac{\epsilon}{3} \tag{1.15a}$$

and

$$\|u(T) - \bar{u}\|_{L^1(\Omega)} \leq \frac{\epsilon}{3}\,. \tag{1.15b}$$

Next, note that the operator F_g in (1.12c) is $L^1(\Omega)$ -contractive:

$$\|F_g u - F_g v\|_{L^1(\Omega)} \leq \|u - v\|_{L^1(\Omega)}\,. \tag{1.16}$$

(1.16) and the contractiveness of $S(t)$ in $L^1(\Omega)$, when combined with the bound (1.15a), imply that the functions computed by (1.12b) satisfy

$$\|u^n(x) - F_{g^*}^{n-n_o} u^{n_o}\|_{L^1(\Omega)} \leq \frac{\epsilon}{3} \tag{1.17}$$

for $n \geq n_o$ and $n_o\tau \geq T_o$. From (A3) with T replaced by $T_o + \tau_o$, it follows that $\exists\, \tau_1$ such that for $0 < \tau \leq \tau_1$

$$\|u^{n_o} - u(n_o\tau)\|_{L^1(\Omega)} \leq \frac{\epsilon}{3} \tag{1.18a}$$

where

$$T_o \leq n_o\tau < T_o + \tau_o\,. \tag{1.18b}$$

Through (1.18) and (1.15b) we bound $\|u^{n_o} - \bar{u}\|_{L^1(\Omega)}$. All we need do is extend this to a bound on $\|F_{g^*}^{n-n_o} u^{n_o} - \bar{u}\|_{L^1(\Omega)}$ and use (1.17) to get the desired result (1.14). To provide the missing link, we observe from (1.11) and (1.12d) that

$$L\, g^*\tau = (I - S(\tau))\bar{g} = L\,(I - S(\tau))f(\bar{u}),$$

and thus

$$\frac{I - S(\tau)}{\tau} f(\bar{u}) = g^*, \tag{1.19}$$

on account of (1.4c) or (1.9b). Hence, from (1.12c),

$$F_{g^*}\bar{u} = \bar{u} \tag{1.20}$$

and

$$\|F_{g^*}^{n-n_0} \; u^{n_0} - \bar{u}\|_{L^1(\Omega)} \leq \|u^{n_0} - \bar{u}\|_{L^1(\Omega)} \quad (1.21)$$

follows from (1.16).

Although (A3) is an assumption, for Ω bounded with smooth boundary and relatively mild conditions on g', it is only a trivial extension of the convergence theorem cited above for the homogeneous problem (1.1) (Ref. 2). For example, sufficient, but certainly not necessary, conditions on g' to establish (A3) would be

$$\int_0^\infty \int_\Omega \left(\sup_{0 \leq y \leq \eta} g'(t+y) - \inf_{0 \leq y \leq \eta} g'(t+y) \right) dx\, dt \to 0 \text{ as } \eta \to 0. \quad (1.22)$$

For Ω unbounded, one may prove (A3) by approximating with arbitrary accuracy in $L^1(\Omega)$, for any $T > 0$, the solutions of (1.12) with $n\gamma \leq T$ by functions which are similarly defined in terms of a boundary value problem with homogeneous boundary data in a suitably large bounded subdomain $\tilde{\Omega}(T) \subset \Omega$.

With regard to the assumptions (A1) or (A2) relating to the approach of solutions of (1.7) or (1.12) to a steady state, we note that these assumptions may be verified directly in particular cases of interest, such as the one-phase Stefan problem in the absence of sources. More generally, when the operator $S(t)$ is strictly contractive in $L^1(\Omega)$ and for some $a > 0$

$$\|S(t)\|_{L^1(\Omega)} \leq e^{-at}, \quad (1.23)$$

an adaptation of the argument which led to (1.16), in which we replace F_g by F_{g^*}, u by u^n, and v by u^{n-1}, enables us to establish (A2). When the assumption of strict contractiveness is dropped, one may still derive a partial result regarding the approach to steady state of the quantities generated by the algorithm (1.12) (cf. Lemma 1 of the next section).

We will show when (A1) - (A3) hold that, under suitable further restrictions on g', the steady state value $\bar{u}$ of the solution of (1.7) coincides with the steady state value $\bar{U}$ of the solution U of the problem

$$U_t + L\, f(U) = \bar{g}, \; x \in \Omega, \; t > 0, \quad (1.24a)$$

$$B\, f(U) = 0, \; x \in \partial\Omega, \; t > 0, \quad (1.24b)$$

$$U(0) = u_0 + \int_0^\infty g'(t)\, dt, \; x \in \Omega. \quad (1.24c)$$

A proof which is general with respect to operators L and functions f with the properties (1.2) and (1.3), but with some restrictions on g' and u_o , will be given in the next section. Our method will be to establish that a similar result holds exactly for the functions generated by the approximate algorithm (1.12), under assumption (A2), and that such a result holds approximately under assumption (A2'). In either case, the property is established for the exact solution by using the convergence of the approximate solution to the exact one, as assumed in (A3). In the third section we will discuss a direct proof of the theorem, without reference to our approximation of the exact solution and the convergence of the approximation, for the special case of a one-phase Stefan problem with $\bar{g} = 0$. This will serve to illuminate the sorts of problems we may expect in trying to obtain a direct proof for the more general case. The fourth section produces, for the case of a one-phase Stefan problem with $\bar{g} = 0$, first a counter-example to one possible generalization of the theorem of Section 2, and then a proof of a weakened form of the generalization. The final section gives an application of the results obtained for the one-phase Stefan problem with $\bar{g} = 0$, whereby it is shown that a time-dependent free boundary problem which arises in the theory of anodic smoothing is equivalent to the steady state of a one-phase Stefan problem.

Before proceeding, let us look at the problems (1.7) and (1.24) whose steady state solutions we compare. In either case, we get equation (1.11) for the steady state. Because of (1.4) or (1.9), $f(\bar{u})$ is thus uniquely determined. Accordingly, if f is 1-1, all our results reduce to a triviality. The only interesting case is when f(u), constrained to be monotone by (1.2), is constant over an interval of values $[u^-, u^+]$ of u. Stefan problems are the best-known problems in this category. With h(x) given as the solution of (1.4a,b) or (1.4a,b) and (1.9a), we will find it convenient to define the functions $u^{\pm}(x)$ by

$$u^+(x) = \sup\{\xi \mid f(\xi) = h(x)\}, \qquad (1.25a)$$

$$u^-(x) \quad \inf\{\xi \mid f(\xi) = h(x)\}. \qquad (1.25b)$$

It is apparent from the outline given above that at the current time our progress regarding the steady state problem has been uneven. This is because the results reported here were originally obtained, in the course of work on water waves (Ref. 9), for the one-phase Stefan problem, and at present they have been extended to the more general problem (1.7) only in part. The incomplete state of this work is, of course, also reflected in the fact that we have found it necessary to assume (A1) - (A3) in order to obtain the principal theorem of this paper, relating the steady states $\bar{u}$ and $\bar{U}$ of (1.7) and (1.24), instead of obtaining the result directly without such assumptions.

Indeed, the restriction of u_o and $\int_0^\infty |g'|dt$ to $L^1(\Omega)$ is probably unnecessary when Ω is unbounded, and it seems quite likely to us that assumption (A2) can be dispensed with and replaced by the premise of the following conjecture.

Conjecture: If, for all u with supp $u \subset \mathfrak{D}$, $\mathfrak{D}$ bounded and $\mathfrak{D} \subset \Omega$, there is a $t_\epsilon(\mathfrak{D}, \tilde{\mathfrak{D}})$ such that, for $\tilde{\mathfrak{D}}$ bounded and $\tilde{\mathfrak{D}} \subset \Omega$,

$$\|S(t)u\|_{L^1(\tilde{\mathfrak{D}})} \leq \epsilon \|u\|_{L^1(\mathfrak{D})}$$

when $t \geqslant t_\epsilon(\mathfrak{D}, \tilde{\mathfrak{D}})$, then

$$\|\max(u_o - u^+, 0) + \max(u^- - u_o, 0)\|_{L^1(\Omega)} < \infty$$

implies that

$$\|\max((F^n_{g^*} u_o) - u^+, 0) + \max(u^- - (F^n_{g^*} u_o), 0)\|_{L^1(\tilde{\mathfrak{D}})} \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

for any bounded domain $\tilde{\mathfrak{D}} \subset \Omega$.

At present we have not satisfied ourselves as to the validity of this conjecture.

As I write this up, it occurs to me that the question of the steady state of (1.7) really involves only two things -- the linear operator L and the function f — and that it should be possible to derive all our results in a more direct and simple way, in a manner which can be extended to the case when u is a vector, solely in terms of properties of the linear operator and the nonlinear algebraic function. However, I leave that to the future.

2. The Case of "Positive Sources"

The main task of this section is to prove the following.

Theorem 1: Under assumptions (A1) - (A3) or (A1), (A2'), (A3), and conditions (1.10) and (1.22) on g, if in addition

$$u_o(x) \geqslant u^-(x),\ x \in \Omega\ , \qquad (2.1a)$$

and

$$g'(x,t) \geqslant 0,\ x \in \Omega\ ,\ t > 0, \qquad (2.1b)$$

the steady state $\overline{u}$ of the solution of (1.7) is the same as the steady state $\overline{U}$ of the solution of (1.24). ($u^-(x)$ is defined by (1.25b).)

We call this the case of "positive sources" because of the inequalities (2.1a) and (2.1b). As we stated in the Introduction, the proof of the theorem will be obtained by referring to the algorithm (1.12) and examining the properties of its steady state, or approximate steady state, according to whether we assume (A2) or (A2'). First, we give a lemma which shows that a quantity closely related to u^n in (1.12) always has a steady state.

Lemma 1: When f satisfies (1.2), g^* is given by (1.12d) with $\overline{g} \in R(L)$, and F_g is given by (1.12c), for $u \in L^1(\Omega)$ the quantities

$$\xi_n \equiv \max(\min(F^n_{g^*}\, u, u^+), u^-) \qquad (2.2a)$$

approach a limit

$$\| \xi_n - \overline{F}_{g^*}\, u \|_{L^1(\Omega)} \longrightarrow 0 \text{ as } n \longrightarrow \infty \quad . \qquad (2.2b)$$

Proof: We use the L^1 -contractiveness of F_{g^*}, as expressed by (1.16), and the fact that $\overline{u} \in f^{-1}(h)$ where h satisfies (1.4a,b) (and (1.9a) where applicable) is a fixed point of F_{g^*}, as expressed by (1.20). Since from (2.2a) and (1.25) $\xi_n \in f^{-1}(h)$,

$$\| F^{n+1}_{g^*}\, u - \xi_n \|_{L^1(\Omega)} \leq \| F^n_{g^*}\, u - \xi_n \|_{L^1(\Omega)} \quad . \qquad (2.3)$$

But

$$\| F^n_{g^*}\, u - \xi_n \|_{L^1(\Omega)} = \| \max(F^n_{g^*}\, u, u^+) - u^+ \|_{L^1(\Omega)} + \| u^- - \min(F^n_{g^*}\, u, u^-) \|_{L^1(\Omega)}$$

and

$$\| F^{n+1}_{g^*}\, u - \xi_n \|_{L^1(\Omega)} = \| \xi_{n+1} - \xi_n \|_{L^1(\Omega)} + \| \max(F^{n+1}_{g^*}\, u, u^+) - u^+ \|_{L^1(\Omega)} + \| u^- - \min(F^{n+1}_{g^*}\, u, u^-) \|_{L^1(\Omega)} .$$

Inserting these expressions into (2.3) and repeating the result for n+1, n+2, ..., n + m - 1, we obtain

$$\sum_{p=1}^{m} \|\xi_{n+p} - \xi_{n+p-1}\|_{L^1(\Omega)} + \|\max(F_{g^*}^{n+m} u, u^+) - u^+\|_{L^1(\Omega)} + \|u^- - \min(F_{g^*}^{n+m} u, u^-)\|_{L^1(\Omega)}$$

$$\leq \|\max(F_{g^*}^{n} u, u^+) - u^+\|_{L^1(\Omega)} + \|u^- - \min(F_{g^*}^{n} u, u^-)\|_{L^1(\Omega)} . \qquad (2.4)$$

(2.2b) follows from (2.4) immediately.

Remark 1: (2.2b) defines the operator $\overline{F}_{g^*}$.

Remark 2: Under assumption (A2), the quantities $F_{g^*}^n u$ approach $\overline{F}_{g^*} u$ as $n \to \infty$. Under assumption (A2'), for $\epsilon > 0$, $n\tau \geq T_0(\epsilon)$, and $\tau \leq \tau_1(\epsilon)$,

$$\|\max(F_{g^*}^{n} u, u^+) - u^+\|_{L^1(\Omega)} + \|u^- - \min(F_{g^*}^{n} u, u^-)\|_{L^1(\Omega)} \leq \epsilon \ . \qquad (2.5)$$

Remark 3: It follows from (1.12c), (1.2), and the monotonicity of S that F_{g^*} is a monotone operator. If $u \geq u^-$, the ξ_n given by (2.2a) satisfy

$$u^+ \geq \xi_{n+1} \geq \xi_n \qquad , \qquad (2.6)$$

and then the convergence expressed by (2.2b) holds also in $L^\infty(\Omega)$ if $u^+ \in L^\infty(\Omega)$.

The next lemma gives our basic result for positive sources.

Lemma 2: When f satisfies (1.2), g* is given by (1.2d) with $\overline{g} \in R(L)$, F_g is given by (1.12c), and $\overline{F}_{g^*}$ is defined by (2.2b), we have, for $u_0 \geq u^-$ and $u_1 \geq 0$,

$$\|\overline{F}_{g^*}(u_0 + u_1) - \overline{F}_{g^*}(F_{g^*} u_0 + u_1)\|_{L^1(\Omega)} = 0. \qquad (2.7)$$

Proof: Let us define, for integers $n \geq 0$,

$$A_n = F_{g^*}^n (u_0 + u_1), \qquad (2.8a)$$

$$A_{n+1/2} = F_{g^*}^n (F_{g^*} u_0 + u_1). \qquad (2.8b)$$

We will show that, for p half of a non-negative integer, we can write

$$A_p = u_0^{(p)} + u_1^{(p)} \qquad (2.9a)$$

and

$$A_{p+1/2} = F_{g^*} u_0^{(p)} + u_1^{(p)} , \qquad (2.9b)$$

where

$$u_0^{(p)} \geq u^- , \ u_1^{(p)} \geq 0. \qquad (2.10)$$

The result is obviously true for p = 0 if we write

$$u_0^{(0)} = u_0 , \; u_1^{(0)} = u_1 . \tag{2.11}$$

If we can prove that there is a decomposition (2.9), (2.10) for $p = 1/2$, a similar proof will establish the result for all suitable p.

We want to show that $\exists \; u_0^{(1/2)} \geq u^-$ such that

$$A_1 - A_{1/2} = F_{g*} \, u_0^{(1/2)} - u_0^{(1/2)} \tag{2.12a}$$

and

$$A_{1/2} \geq u_0^{(1/2)} . \tag{2.12b}$$

From (1.12c),

$$A_1 - A_{1/2} = (S - I)\, f(u_0 + u_1) - (S - I)\, f(u_0) \tag{2.13a}$$

and from (1.12c), (1.19), (1.11), (1.25), and (1.4a,b) (and (1.9a) if appropriate),

$$F_{g*} \, u_0^{(1/2)} - u_0^{(1/2)} = (S - I)\,(f(u_0^{(1/2)}) - f(u^-)). \tag{2.13b}$$

We can make the right-hand sides of (2.12a) and (2.12b) equal if we choose

$$u_0^{(1/2)} = \inf \{ \xi \mid f(\xi) = f(u_0 + u_1) - f(u_0) + f(u^-) \} . \tag{2.14}$$

Since $u_1 \geq 0$, it follows from (2.14) and (1.25b) that $u_0^{(1/2)} \geq u^-$. To establish (2.12b), we find

$$\begin{aligned} f(A_{1/2}) &= f(u_0 + u_1 + (S - I)\,(f(u_0) - f(u^-))) \\ &\geq f(u_0 + u_1 - f(u_0) + f(u^-)) \\ &\geq f(u_0 + u_1) - f(u_0) + f(u^-) = f(u_0^{(1/2)}), \end{aligned} \tag{2.15}$$

upon use of the contractiveness of S in $L^\infty(\Omega)$ (monotonicity), $u_0 \geq u^-$, the property (1.2) of f, and (2.14). Comparing with (2.14), we get the desired result. Hence we may regard (2.9) and (2.10) as established.

From Lemma 1, in $L^1(\Omega)$ $\quad \min(A_n, u^+) \longrightarrow \overline{F}_{g*}\,(u_0 + u_1)$ as $n \to \infty$.

Thus,

$$\int_{A_n \leq u^+} \left[\min(F_{g*} A_n , u^+) - \min(F_{g*} \min(A_n , u^+), u^+)\right] dx \longrightarrow 0 \text{ as } n \longrightarrow \infty . \tag{2.16}$$

As was pointed out in Remark 3, F_{g*} is monotone. So is the operator $\min(F_{g*}(\cdot), u^+)$. Thus, if w_n is a function satisfying

$$\min(A_n , u^+) \leq w_n \leq A_n , \tag{2.17a}$$

it follows from (2.16) that

$$\int_{w_n < u^+} \left[\min(F_{g*} w_n , u^+) - w_n\right] dx \longrightarrow 0 \text{ as } n \longrightarrow \infty , \tag{2.17b}$$

or, with reference to (1.12c),

$$\int_{w_n<u^+}(F_{g^*}w_n - w_n)\,dx = \int_{w_n<u^+}\left[(S - I)\, f(w_n) + g^*\gamma\right]dx \to 0 \text{ as } n \to \infty \quad . \tag{2.17c}$$

Pick

$$w_n = \begin{cases} u_o^{(n)} + u_1^{(n)} & u_o^{(n)} + u_1^{(n)} < u^+ \\ u^+ & u_o^{(n)} < u^+,\ u_o^{(n)} + u_1^{(n)} \geq u^+ . \\ u_o^{(n)} & u_o^{(n)} \geq u^+ \end{cases} \tag{2.18}$$

It is clear that w_n satisfies (2.17a). In addition

$$f(w_n) = f(u_o^{(n)}), \tag{2.19}$$

on account of (2.10). Accordingly, from (2.17c),

$$\int_{A_n<u^+}\left[(S - I)\, f(u_o^{(n)}) + g^*\gamma\right]dx = \int_{A_n<u^+} \left|F_{g^*}u_o^{(n)} - u_o^{(n)}\right|dx \to 0 \text{ as } n \longrightarrow \infty \ . \tag{2.20}$$

Where $A_n = u_o^{(n)} + u_1^{(n)} \geq u^+$, the monotonicity of F_{g^*} implies that

$$F_{g^*}u_o^{(n)} + u_1^{(n)} \geq \min(u_o^{(n)}, u^+) + u_1^{(n)} \geq u^+, \tag{2.21}$$

or $A_{n+1/2} \geq u^+$, from (2.9b). Integrating this result over the set $\{x \mid A_n(x) \geq u^+\}$ and adding it to (2.20) we get, on referring to (2.9),

$$\int \left|\min(A_{n+1/2}, u^+) - \min(A_n, u^+)\right|dx \longrightarrow 0 \text{ as } n \to \infty \ . \tag{2.22}$$

Recalling the definitions of A_n and $A_{n+1/2}$ in (2.8), we see that the lemma is proved.

Upon repeated application of (2.7) we get, when $u_o \geq u^-$ and $u_i \geq 0$, $1 \leq i \leq m$,

$$\begin{aligned} &\overline{F}_{g^*}(F_{g^*}^{m} u_o + F_{g^*}^{m-1} u_1 + \dots + F_{g^*} u_{m-1} + u_m) \\ &= \overline{F}_{g^*}(F_{g^*}^{m-1} u_o + F_{g^*}^{m-2} u_1 + \dots + F_{g^*} u_{m-2} + u_{m-1} + u_m) \\ &= \overline{F}_{g^*}\left(\sum_{i=0}^{m} u_i\right) \text{ in } L^1(\Omega) \quad . \end{aligned} \tag{2.23}$$

To prove Theorem 1, all we need to do is make several observations. First, from the convergence assumption (A3) and the L^1-contractiveness of F_{g^*}, it follows that the solutions of (1.7) and (1.24) obey a stability relation with respect to changes in the initial data and inhomogeneous terms: If

$$\tilde{u}_t + L\, f(\tilde{u}) = \tilde{g}(t),\ x \in \Omega\ ,\ t > 0, \tag{2.24a}$$

$$B\, f(\tilde{u}) = 0,\ x \in \partial\Omega\ ,\ t > 0, \tag{2.24b}$$

and

$$\tilde{u}(0) = \tilde{u}_o\ ,\ x \in \Omega\ , \tag{2.24c}$$

then

$$\|\tilde{u}(x,t) - u(x,t)\|_{L^1(\Omega)} \leq \|\tilde{u}_o - u_o\|_{L^1(\Omega)} + \int_o^t \|\tilde{g}(t') - g(t')\|_{L^1(\Omega)} dt' \ . \quad (2.24d)$$

Second, using this and properties (1.10) and (1.22) of g, we see that for τ sufficiently small and $m\tau$ sufficiently large, the solution U of (1.24) may be replaced with arbitrary accuracy in $L^1(\Omega)$ by the solution of

$$\tilde{U}_t + L\, f(\tilde{U}) = \bar{g},\ x \in \Omega\ ,\ t > 0, \quad (2.25a)$$

$$B\, f(\tilde{U}) = 0,\ x \in \partial\Omega\ ,\ t > 0, \quad (2.25b)$$

$$\tilde{U}(0) = u_o + \sum_{n=1}^{m} g'(n\tau)\tau,\ x \in \Omega\ . \quad (2.25c)$$

Third, we use (1.10), (1.22), (1.12b), the density of D(L) in $L^1(\Omega)$, the fact that, for $u \in D(L)$,

$$(S(\tau) - I)\, u \longrightarrow 0 \text{ as } \tau \longrightarrow 0, \quad (2.26)$$

and the L^1-contractiveness of F_{g*}, to show that for τ sufficiently small and $m\tau$ sufficiently large, the functions u^n of (1.12) may be replaced with arbitrary accuracy in $L^1(\Omega)$ by the functions $\tilde{u}^n$ given by

$$\tilde{u}^o = u^o, \quad (2.27a)$$

$$\tilde{u}^{n+1} = F_{g*}\tilde{u}^n + \begin{cases} g'((n+1)\tau)\tau & n < m \\ 0 & n \geq m \end{cases} . \quad (2.27b)$$

From the assumed convergence (A3) and the assumption that a steady state is approached, (A2) or (A2'), we note that the steady state $\bar{\tilde{U}}$ of the solution of (2.25) can be approximated by the functions $\tilde{U}^n$ given by

$$\tilde{U}^o = u_o + \sum_{p=1}^{m} g'(p\tau)\tau,\ x \in \Omega\ , \quad (2.28a)$$

$$\tilde{U}^{n+1} = F_{g*}\tilde{U}^n, \quad (2.28b)$$

with arbitrary accuracy in $L^1(\Omega)$ if $n\tau$ is sufficiently large and τ is sufficiently small; similarly the steady state $\bar{u}$ of the solution of (1.7) can be approximated with arbitrary accuracy in $L^1(\Omega)$ for $n\tau$ large enough and τ small enough by the functions $\tilde{u}^n$ generated in (2.27).

When $u_o \geq u^-$ and $g' \geq 0$, (2.23) shows that $\tilde{u}^n$ and $\tilde{U}^n$ just given satisfy

$$\|\min(\tilde{U}^n, u^+) - \min(\tilde{u}^n, u^+)\|_{L^1(\Omega)} \longrightarrow 0 \text{ as } n \rightarrow \infty\ . \quad (2.29)$$

Finally, when we combine the above observations with Remark 2, we obtain Theorem 1 easily.

3. Direct Proof for the One-Phase Stefan Problem

It would be interesting to see to what extent a direct proof of Theorem 1 can be fashioned, one which has no recourse to the algorithm (1.12) and the approximate solution which it generates. We will find it convenient to introduce, besides the dependent function u, the function v(x) defined by

$$v(x) \quad = \quad \int_0^\infty (f(u(x,t)) - h(x))\,dt \quad , \qquad (3.1)$$

where h(x) satisfies (1.4a,b) and (1.9a) when appropriate. From (1.7), (1.10) and (A1),

$$L\,v \quad = \quad \int_0^\infty g'(t)\,dt \quad + \quad u_o \; - \bar{u}. \qquad (3.2)$$

From (1.4c) or (1.9b), v given by (3.2), when it exists, will be unique a.e. Theorem 1 says that $\bar{u}$ in (3.2) depends on u_o and g' only in the combination

$$u_o \; + \; \int_0^\infty g'(t)\,dt$$

when $u_o \geqslant u^-$ and $g' \geqslant 0$. Thus, v also depends on u_o and g' only through that combination.

Let us consider the problem of determining v for the one-phase Stefan problem. This problem is the special case of (1.7) with $L \; -\Delta$, $\bar{g} = 0$, and

$$f(u) \quad = \quad \begin{cases} 0 & u \leq \lambda \\ u - \lambda & u \geqslant \lambda \end{cases} \quad . \qquad (3.3)$$

(We must have $\lambda \geqslant 0$ in order to satisfy $f(0) = 0$.) From (1.25) we have $u^+(x) = \lambda$, $u^-(x) = -\infty$, and thus the condition (2.1a) is satisfied automatically.

A typical set of initial conditions is the following:

$$u_o \geq \lambda \quad , \; x \in \Omega_o \subset \Omega \qquad , \qquad (3.4a)$$

$$u_o = 0, \; x \in \Omega - \Omega_o \;. \qquad (3.4b)$$

Since we are not looking for maximum generality, we consider the case where the sources satisfy

$$\operatorname{supp} g'(\cdot\,,t) \;\subset\; \mathrm{Cl}(\,\Omega_o) \quad \forall \quad t \geqslant 0. \qquad (3.4c)$$

The problem always has a steady state, since, when $g' \geqslant 0$, the set

$$\Omega\,(t) \equiv \mathrm{Int}\left\{ x \mid u(x,t) \geqslant \lambda \right\} \qquad (3.5a)$$

increases monotonically with t and

$$\operatorname{meas}(\Omega(t)) \leq \frac{1}{\lambda}\left(\int u_0 dx + \iint g' dx dt\right) \qquad \forall\ t. \tag{3.5b}$$

The steady state is of the form

$$\bar{u} = \begin{cases} \lambda & x \in \bar{\Omega} \\ 0 & x \in \operatorname{Int}(\Omega - \bar{\Omega}) \end{cases}, \tag{3.6a}$$

where the set $\bar{\Omega}$ satisfies

$$\Omega_0 \subset \bar{\Omega} \subset \Omega \quad . \tag{3.6b}$$

From (3.1),

$$v(x) = \int_{t^*(x)}^{\infty} (u(x,t) - \lambda)\, dt \tag{3.7a}$$

where

$$t^*(x) \equiv \inf\{\xi \mid u(x,\xi) \geq \lambda\} \quad . \tag{3.7b}$$

Clearly

$$v(x) > 0,\ x \in \operatorname{Int}(\bar{\Omega}), \tag{3.8a}$$

$$v(x) = 0,\ x \in \Omega - \operatorname{Int}(\bar{\Omega}), \tag{3.8b}$$

and in particular

$$v(x) = 0,\ x \in \partial\bar{\Omega} - \partial\Omega \quad . \tag{3.8c}$$

From (3.7a) and the Stefan equation,

$$\int_{\partial\Omega(t)-\partial\Omega} - n\cdot\nabla v\, dS = \int_{(\partial\bar{\Omega}-\partial\Omega(t))\cap\partial\Omega} - n\cdot\nabla v\, dS + \lambda \operatorname{meas}(\bar{\Omega} - \Omega(t)), \tag{3.9a}$$

where in the first integral n points out of $\Omega(t)$, and in the second n points out of Ω . This gives us

$$n\cdot\nabla v(x) = 0,\ x \in \partial\bar{\Omega} - \partial\Omega \quad . \tag{3.9b}$$

According to (3.6), inside $\bar{\Omega}$ v satisfies

$$\Delta v = \lambda - u_0 - \int_0^{\infty} g'(t)\, dt \quad . \tag{3.10}$$

Boundary conditions on $v(x)$ for $x \in \partial\Omega$ are found from (1.1b):

$$B\, v = 0,\ x \in \partial\Omega \quad . \tag{3.11}$$

If we can show that v depends on u_0 and g' only through the combination

$$U_0 = u_0 + \int_0^{\infty} g'\, dt \quad , \tag{3.12}$$

it will follow from (3.2) that $\bar{u}$ is also dependent only on U_0 . But (3.10), (3.8c), (3.9b), (3.11), (3.9a), and (3.6b) define an elliptic free boundary

problem, and the question becomes one of the uniqueness of its solution. This elliptic free boundary problem is of a type which has been studied in connection with a diffusion-consumption problem (Refs. 1, 8). With v extended to $\Omega - \bar{\Omega}$ according to (3.8b), the problem may be written as the determination of the steady state of

$$w_t = \Delta w + U_o - \begin{cases} \lambda & w > 0 \\ 0 & w = 0 \end{cases}, \qquad (3.13a)$$

$$w(0) = w_o \geqslant 0. \qquad (3.13b)$$

That is, any solution of the elliptic free boundary problem will also be a steady state solution of (3.13) for an appropriate choice of w_o. (Recall that we have assumed supp $U_o \subset \mathrm{Cl}(\Omega_o)$.) By regarding (3.13) as the limit as $\epsilon \downarrow 0$ of a problem of the form

$$w^{\epsilon}_t = \Delta w^{\epsilon} + U_o - \lambda \begin{cases} 1 & w^{\epsilon} \geqslant \epsilon \\ \dfrac{w^{\epsilon}}{\epsilon} & w^{\epsilon} \leq \epsilon \end{cases}, \qquad (3.14)$$

one may prove the monotone dependence of the steady state on w_o (Ref. 8). In particular, from (3.8a) and (3.2), the steady state will satisfy

$$v^- \leq v \leq v^+ \qquad (3.15)$$

where $v^{\mp}$ are the steady state solutions of (3.13) corresponding respectively to

$$w_o^- = 0 \qquad (3.16a)$$

and

$$w_o^+ = \int_{\Omega_o} G(x,x')\,(U_o(x') - \lambda)\,dx' \qquad (3.16b)$$

where

$$\Delta G(x,x') = -\delta(x - x'),\ x \in \Omega,\ x' \in \Omega, \qquad (3.16c)$$

$$B\,G(x,x') = 0,\ x \in \partial\Omega,\ x' \in \Omega. \qquad (3.16d)$$

(We can show that

$$w^-(x,t) = \int_0^t f(u(x,t'))\,dt'. \qquad (3.17))$$

The monotonicity cited for solutions of (3.13) shows that the respective solutions $w^{\mp}$ corresponding to the initial data $w_o^{\mp}$ satisfy

$$0 \leq w^+ - w^- \leq \gamma \qquad (3.18a)$$

where γ satisfies

$$\gamma_t = \Delta \gamma \quad , \tag{3.18b}$$

$$\gamma(0) = w_o^+ \quad , x \in \Omega \, , \tag{3.18c}$$

$$B\gamma = 0, x \in \partial\Omega \, . \tag{3.18d}$$

By using (1.4c) or (1.9b), we find that the steady state of γ is 0, and thus from (3.15)

$$v^- = v = v^+, \tag{3.19}$$

that is, the solution of the elliptic free boundary problem is unique.

4. Positive and Negative Sources in the One-Phase Stefan Problem

In proving Theorem 1, we assumed that the sources g' were non-negative. In the case of the one-phase Stefan problem where $L \; -\Delta$, $\bar{g} = 0$, and f is given by (3.3), we can show that the theorem does not generally hold without imposing the condition (2.1b). For example, one may have U_o given by (3.12) such that

$$U_o \geq \lambda \quad , x \in \Omega_o \subset \Omega \quad , \tag{4.1a}$$

$$U_o = 0, x \in \Omega - \Omega_o \quad , \tag{4.1b}$$

and also g' such that

$$g' \leq 0 \quad \forall \quad x \in \Omega \quad \text{and} \quad \forall \quad t > 0, \quad \int g' dt < 0, \; x \in \Omega_1 \subset \Omega_o \, . \tag{4.1c}$$

In this case the steady state of the solution of (1.24) will have $\bar{U} = \lambda$ for $x \in \Omega_o$. However, suppose $g'(x,t) = 0$ for $t < T_o$. If T_o is large enough, the solution of (1.7) will approach, as $t \longrightarrow T_o$, a function which is $< \lambda + \epsilon$ everywhere, where $\epsilon > 0$ and $\epsilon \longrightarrow 0$ as $T_o \longrightarrow \infty$. Then the addition of the sources g' for $T_o < t < \infty$ will result finally in a steady state $\bar{u}$ which satisfies $\bar{u} < \lambda$ for $x \in \Omega_1$. Clearly this is not the same as $\bar{U}$.

The proof given in the last section of the uniqueness of v satisfying (3.10), (3.8c), (3.9b), (3.11), (3.8a), and (3.6b) remains valid even in the case of negative sources. However, when the requirement $g' \geq 0$ is dropped, the monotonicity expressed by (3.5a) no longer holds, and the steady state $\bar{u}$ need not be given by (3.6a). Accordingly, the equation (3.2) satisfied by v will not

lead to (3.10) in general, but only when $\overline{u} = \lambda$ where $v > 0$. Since $v > 0$ at a point x means that at some time t, $u(x,t) \geq \lambda$, we can insure that $\overline{u}$ will satisfy this requirement if for all $x \in \Omega$ and $t \geq 0$ we have

$$\int_t^\infty g'(x,t')dt' \geq 0. \qquad (4.2)$$

More generally, let u and $\tilde{u}$ be solutions of the one-phase Stefan problem with initial data u_0, $\tilde{u}_0$ and sources g', $\tilde{g}'$, respectively, such that

$$U_0 = u_0 + \int_0^\infty g'\,dt = \tilde{u}_0 + \int_0^\infty \tilde{g}'\,dt \;, \qquad (4.3)$$

and let

$$v = \int_0^\infty f(u(x,t))dt \;, \; \tilde{v} = \int_0^\infty f(\tilde{u}(x,t))dt \;. \qquad (4.4)$$

Define

$$\sigma \equiv \left\{ x \in \Omega \mid v(x) > 0 \text{ and } \int_t^\infty g'(x,t')dt' < 0 \text{ for some } t \geq 0 \right\}, \qquad (4.5a)$$

$$\tilde{\sigma} \equiv \left\{ x \in \Omega \mid \tilde{v}(x) > 0 \text{ and } \int_t^\infty g'(x,t')dt' < 0 \text{ for some } t \geq 0 \right\}. \qquad (4.5b)$$

If

$$\overline{u}(x) = \overline{\tilde{u}}(x) \text{ for all } x \in \sigma \cup \tilde{\sigma} \;, \qquad (4.6a)$$

then

$$\overline{u}(x) = \overline{\tilde{u}}(x) \text{ for all } x \in \Omega \;. \qquad (4.6b)$$

This result is derived easily from the following observations. First, if $x \in \Omega - \sigma$, either $v(x) = 0$ or $v(x) > 0$ and $\int_t^\infty g'(x,t')\,dt' \geq 0$ for all $t \geq 0$. In the latter case, $u(x,t) > \lambda$ for some $t \geq 0$, and the condition on g' implies that $\overline{u}(x) = \lambda$.

Second, v satisfies the following elliptic free boundary problem:

$$v(x) > 0 \text{ for } x \in \mathrm{Int}(\overline{\Omega}), \qquad (4.7a)$$

$$\overline{\Omega} \supset \left\{ x \,\middle|\, u_0 + \int_0^\infty g'\,dt \geq \lambda \right\} \;, \qquad (4.7b)$$

$$v(x) = 0, \; x \in \partial\overline{\Omega} - \partial\Omega \;, \qquad (4.7c)$$

$$n \cdot \nabla v(x) = 0, \; x \in \partial\overline{\Omega} - \partial\Omega \;, \qquad (4.7d)$$

$$B\,v = 0, \; x \in \partial\Omega \;, \qquad (4.7e)$$

$$\Delta\, v = \overline{u} - U_0 \;, \; x \in \mathrm{Int}(\overline{\Omega}). \qquad (4.7f)$$

Since we only compare solutions of (4.7) for which $\overline{u}$ takes on the same given prescribed value at some points of $\overline{\Omega}$ and is λ over the rest of $\overline{\Omega}$, the uniqueness of v follows as before. By (3.2), the uniqueness of $\overline{u}$ is estab-

lished under these conditions. Note that we no longer have the restrictions (3.4) on u_o and g'.

It is also possible to verify these facts by direct consideration of the approximating algorithm (1.12) and invocation of the convergence of the approximate solution to the exact solution.

5. Application to a Problem of Anodic Smoothing

Consider the Stefan problem

$$u_t = \Delta f(u) \tag{5.1}$$

where f(u) is given by (3.3) and with initial data

$$u_o = \lambda + d\alpha\, \xi_o \quad , x \in \Omega_o \subset \Omega \quad , \tag{5.2a}$$

$$u_o = 0,\ x \in \Omega - \Omega_o \quad , \tag{5.2b}$$

where $\xi_o \geqslant 0$ and $d\alpha$ is a small positive number. Let us impose the boundary condition

$$f(u) = 0,\ x \in \partial\Omega \ . \tag{5.2c}$$

For this problem we use the notation

$$dv = \int_o^\infty f(u(x,t))\,dt \ . \tag{5.3}$$

(3.10) becomes

$$\Delta(dv) = -d\alpha\, \xi_o\ ,\ x \in \Omega_o\ , \tag{5.4a}$$

and to lowest order in $d\alpha$, from (3.8c) and (3.9b),

$$dv = 0,\ x \in \partial\Omega_o - \partial\Omega \ . \tag{5.4b}$$

In addition,

$$dv = 0,\ x \in \partial\Omega \ . \tag{5.4c}$$

Thus,

$$n\cdot\nabla(dv) = d\alpha \int_{\Omega_o} n\cdot\nabla G_o(x,x')\ \xi_o(x')\,dx' \ , \tag{5.5}$$

where G_o is given by

$$\Delta G_o(x,x') = -\delta(x-x'),\ x \in \Omega_o\ ,\ x' \in \Omega_o\ , \tag{5.6a}$$

$$G_o(x,x') = 0,\ x \in \partial\Omega_o \cap \partial\Omega \quad , x' \in \Omega_o\ , \tag{5.6b}$$

$$G_o(x,x') = 0,\ x \in \partial\Omega_o - \partial\Omega \quad , x' \in \Omega_o\ . \tag{5.6c}$$

We denote the steady state domain $\overline{\Omega}$ by

$$\bar{\Omega} = \Omega_{d\alpha} \quad . \tag{5.7}$$

From (3.9b) and (3.10), (5.5) implies that $\Omega_{d\alpha}$ is obtained from Ω_o by traveling outward normal to Ω_o at each point of $\partial\Omega_o - \partial\Omega$ a distance

$$-\frac{1}{\lambda}\, d\alpha \int_{\Omega_o} n\cdot\nabla G_o(x,x')\ \xi_o(x')\,dx' \quad .$$

We may now envisage a collection of such problems, in which we generate domains $\Omega_\alpha \subset \Omega$, $\alpha > 0$, where $\Omega_{\alpha+d\alpha} \supset \Omega_\alpha$ is found from Ω_α by traveling outward along the normal to $\partial\Omega_\alpha$ at each point of $\partial\Omega_\alpha - \partial\Omega$ a distance

$$-\frac{1}{\lambda}\, d\alpha\ n\cdot\nabla v_\alpha \equiv V(\alpha)\,d\alpha \ , \tag{5.8a}$$

where

$$\Delta\, v_\alpha = -\xi_\alpha \qquad , x \in \Omega_\alpha \ , \tag{5.8b}$$

$$v_\alpha = 0,\ x \in \partial\Omega_\alpha \ , \tag{5.8c}$$

and

$$\xi_\alpha \geq 0,\ x \in \Omega_\alpha \ , \tag{5.9a}$$

$$\xi_\alpha = 0,\ x \in \Omega - \Omega_\alpha \quad . \tag{5.9b}$$

This collection of problems may be thought of as defining a "time-dependent" free boundary problem, where α represents the "time" variable.

The results of Sections 3 and 4 show that the region Ω_α can be obtained directly as the support of the steady state $\bar{u}$ of the solution of (5.1) with initial data

$$u_o = \lambda + \int_0^\alpha \xi_{\alpha'}\, d\alpha' \qquad , x \in \Omega_o \ , \tag{5.10a}$$

$$u_o = \int_0^\alpha \xi_{\alpha'}\, d\alpha' \qquad , x \in \Omega - \Omega_o \quad . \tag{5.10b}$$

The "time-dependent" free boundary problem just described is very similar to a problem which occurs in the theory of anodic smoothing (Refs. 3, 5, 6, 7). To bring problem (5.8) into the form of the anodic smoothing problem, we find a function Θ_α with the following properties:

$$\Theta_\alpha\ (x) = 1,\ x \in \partial\Omega \cap \partial\Omega_\alpha \quad , \tag{5.11a}$$

$$\Theta_\alpha\ (x) = 0,\ x \in \Omega - \Omega_\alpha \quad , \tag{5.11b}$$

$$\Delta\Theta_\alpha \text{ bounded and non-negative, } x \in \Omega \ . \tag{5.11c}$$

Next let

$$\xi_\alpha = \Delta\theta_\alpha \quad . \tag{5.12}$$

With v_α given by (5.8), the function

$$\psi_\alpha = v_\alpha + \theta_\alpha \tag{5.13}$$

has the following properties:

$$\Delta\psi_\alpha = 0, \ x \in \Omega_\alpha , \tag{5.14a}$$

$$\psi_\alpha = 1, \ x \in \partial\Omega \cap \partial\Omega_\alpha \quad , \tag{5.14b}$$

$$\psi_\alpha = 0, \ x \in \partial\Omega_\alpha - \partial\Omega \quad , \tag{5.14c}$$

and Ω_α is generated by traveling outward along the normal to $\partial\Omega_{\alpha'}$, for $0 \le \alpha' < \alpha$, at each point of $\partial\Omega_{\alpha'} - \partial\Omega$, at the rate (with respect to α')

$$v_{\alpha'} = -\frac{1}{\lambda} \ n \cdot \nabla \psi_{\alpha'} \quad . \tag{5.14d}$$

In the anodic smoothing problem (Ref. 3), Ω is the domain exterior to a cathode c, $\partial\Omega_0 \cap \partial\Omega = \partial\Omega_\alpha \cap \partial\Omega = \partial c$ for $\alpha \geq 0$, and $\lambda = 1$. Thus we see from (5.11) that θ_0 may also be used as θ_α, $\alpha > 0$, and from (5.12) ξ_0 may be used as ξ_α for $\alpha > 0$. In that case the initial data for an equivalent steady state Stefan problem are

$$u_0 = \lambda + \alpha\xi_0 \qquad , x \in \Omega_0 , \tag{5.15a}$$

$$u_0 = 0, \ x \in \Omega - \Omega_0 \quad . \tag{5.15b}$$

It also follows that any effective methods for solving the steady state Stefan problem (5.1) and (5.15) (Ref. 2) can be used to solve the anodic smoothing problem. In addition, regularity results for the free surface in the steady state Stefan problem carry over immediately to the free surface for the anodic smoothing problem. Independently, Charles M. Elliott has made similar observations about the equivalence of the anodic smoothing problem and a steady state Stefan problem (Ref. 4).

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